Recall. Over $\mathbb{Z}_{(p)}$. (1)

$$\left.\begin{aligned} \mathrm{Fix} &= (\mathbb{W}_{p^\infty})^{F=\mathrm{id}} \\ \mathrm{Ker} &= (\mathbb{W}_{p^\infty})^{F=0}. \end{aligned}\right\} \text{group schemes.}$$

Goals: (a) Compute $R\Gamma(B\mathrm{Ker}, \mathcal{O})$.

(b) Prove that $QCoh(B\mathrm{Fix}) \cong QCoh(B\mathrm{Ker})$ equivalence of categories.

(Not equivalence of monoidal categories).

Recall: B Bix is an affine stack and infact is the affinization of S^1. ~~over~~

$B\mathrm{Ker}$ is an affine stack too.

(a') $B\mathrm{Ker} = \mathrm{Spec}^{\Delta}(\mathbb{Z}_{(p)}[\eta])$.

where $\mathbb{Z}_{(p)}[\eta]$ is the trivial square zero extension ~~of~~ $\mathbb{Z}_p \oplus \mathbb{Z}_p[-1]$.
$\in$ coSCR.

$\exists$ $\mathbb{G}_m$ equivariant map of group schemes

$\mathrm{Ker} \longmapsto \mathbb{G}_a \rightsquigarrow B\mathrm{Ker} \to B\mathbb{G}_a$

corresponds to an element $u \in H^1(R\Gamma(B\mathrm{Ker}, \mathcal{O}))$

$u : \underset{\text{weight 1}}{\mathbb{Z}_p[-1]} \longmapsto R\Gamma(B\mathrm{Ker}, \mathcal{O})$.

$1 : \underset{\text{weight 0}}{\mathbb{Z}_p} \longmapsto R\Gamma(B\mathrm{Ker}, \mathcal{O})$ (using the structure map $B\mathrm{Ker} \mapsto \mathrm{Spec}\,\mathbb{Z}_{(p)}$)

+ $\mathbb{Z}_p \oplus \mathbb{Z}_p[-1] \longrightarrow R\Gamma(BKer, \mathcal{O})$ ②

WTS this is an equivalence.

a) $(\cdot) \otimes \mathbb{Q}$ is an equivalence, since $Ker \otimes \mathbb{Q} \cong \mathbb{G}_{a,\mathbb{Q}}$

We show

b) $(\ ) \otimes \mathbb{F}_p$ an equivalence as well.

<u>Proof uses cartier duality</u>.

$$Ker = \varprojlim \left(\mathbb{W}_n\right)^{F=0} := \varprojlim Ker^n$$

Ker^n ~~$\Gamma(Ker^n, \mathcal{O})$~~ has cartier dual $\alpha_{p^n}\left\{ := \mathrm{Spec}\ \frac{\mathbb{F}_p[X]}{X^{p^n}}\right\}$

Now $\Gamma(Ker^n, \mathcal{O}) \overset{\text{duality}}{\rightsquigarrow} \frac{\mathbb{F}_p[X]}{X^{p^n}}$

$\Gamma(Ker^n, \mathcal{O})$ co-algebra $\rightsquigarrow$ algebra

$\Gamma(\mathbb{Z}/p^n, \mathcal{O}) \overset{\text{duality}}{\leftsquigarrow} \frac{\mathbb{F}_p[X]}{X^{p^n}-1}$ (a hopf algebra as well (functor of points))

co-algebra. $\leftsquigarrow$ algebra

colimits

$$\Rightarrow \Gamma(Ker, \mathcal{O}) \underset{\text{co-algebra}}{\cong} \Gamma(\mathbb{Z}_p, \mathcal{O}) \ *$$

$$\Rightarrow R\Gamma(BKer, \mathcal{O}) \overset{\text{descent}}{\simeq} \varprojlim \left(\mathbb{F}_p \rightrightarrows \Gamma(Ker, \mathcal{O}) \substack{\to\\\to\\\to} \Gamma(Ker, \mathcal{O})^{\otimes 2} \cdots\right)$$

co-alg. $\|$ co-alg $\|$

$$R\Gamma(\mathbb{Z}_p, \mathcal{O}) \overset{\text{descent}}{\simeq} \varprojlim \left(\mathbb{F}_p \rightrightarrows \Gamma(\mathbb{Z}_p) \substack{\to\\\to\\\to} \Gamma(\mathbb{Z}_p)^{\otimes 2} \cdots\right)$$

$\Rightarrow$ $\mathrm{CoMod}_{\Gamma(\mathrm{Ker}, \mathcal{O})} \cong \mathrm{CoMod}_{\Gamma(B\mathbb{Z}_p, \mathcal{O})}$ (*)

$\Rightarrow$ $\mathrm{QCoh}(B\mathrm{Ker}) \cong \mathrm{QCoh}(B\mathbb{Z}_p)$.

$B\mathbb{Z}_p$ is the affinization of S^1 over $\mathbb{F}_p$

$\Rightarrow$ $R\Gamma(B\mathrm{Ker}, \mathcal{O}) \simeq R\Gamma(B\mathbb{Z}_p, \mathcal{O})$

$\cong \mathbb{F}_p \oplus \mathbb{F}_p[-1]$ (as modules)

Thus $R\Gamma(B\mathrm{Ker}, \mathcal{O}) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p[-1]$ (as modules).

(*) $B\mathrm{Ker} \longmapsto \mathrm{Spec}\,\mathbb{Z}_p$ has finite coh. dim

($\otimes \mathbb{F}_p$, follows from (*) since $B\mathbb{Z}_p$ has finite coh. dimension.

$\otimes \mathbb{Q}$ follows from $B\mathrm{Ker} \otimes \mathbb{Q} = B\mathbb{G}_{a,\mathbb{Q}}$.

Scanned with CamScanner

(a') $R\Gamma(B\mathbb{G}_a, \mathcal{O})$ as a ring.

Step 1 as a graded E_∞-ring $R\Gamma(B\mathbb{G}_a, \mathcal{O})$ is the trivial sq. zero ext^n.

Proof: Let $S := A \oplus A[-1]$.

Then by grading argument

$\Rightarrow$ graded E_∞-A-alg structure on S

$\simeq$ graded A-alg. structure on the graded Module

$$\underset{0}{A} \oplus \underset{1}{A}$$

$=$ Square zero ext^n.

Lemma: Let G be a flat affine group scheme over $\mathbb{Z}_{(p)}$ and assume

(i) BG is of finite coh. dimension.

(ii) $G|_{\mathbb{Q}}$ + $G|_{\mathbb{F}_p}$ are unipotent.

Then $QCoh(BG) \simeq Mod_{R\Gamma(BG,\mathcal{O})}$

Proof: ETS $QCoh(BG)$ is compactly generated by $\mathcal{O}_{BG}$.

(i) $\Rightarrow$ $\mathcal{O}_{BG}$ is compact.

To show generators need to show

if $Map(\mathcal{O}, E) \simeq 0 \Rightarrow E = 0$.

ETS

$f: BG \to \operatorname{Spec} \mathbb{Z}_p \qquad f_* E = 0 \Rightarrow E = 0$

$$\begin{tikzcd}
\operatorname{Spec}\mathbb{Q} \arrow[r, "j"] \arrow[d, shift left] & \operatorname{Spec}\mathbb{Z}_p \arrow[d, shift right] & \operatorname{Spec}\mathbb{F}_p \arrow[l, "i"'] \arrow[d, shift right] \\
BG|_{\mathbb{Q}} \arrow[u, shift left, "f_{\mathbb{Q}}"] \arrow[r, "J"] & BG \arrow[u, shift right, "f"'] & BG_{\mathbb{F}_p} \arrow[l, "I"'] \arrow[u, shift right, "f_p"']
\end{tikzcd}$$

$\mathbb{F}_p$

$$f_* E \otimes_{\mathbb{Z}_p} \mathbb{F}_p \simeq (f_p)_* I^* E = RHom(\mathcal{O}, I^* E)$$

$$f_* E \otimes_{\mathbb{Q}} \mathbb{F}_p \simeq (f_{\mathbb{Q}})_* J^* E = RHom(\mathcal{O}, J^* E)$$

Since $G|_{\mathbb{Q}}$ + $G|_{\mathbb{F}_p}$ are unipotent, $\mathcal{O}$ is a generator in $BG|_{\mathbb{Q}}$ and $BG|_{\mathbb{F}_p}$

$$\Rightarrow \quad I^* E,\ J^* E = 0$$

$$\Rightarrow \quad E = 0$$

Thus $\quad QCoh(BFix) \simeq Mod_{R\Gamma(BFix, \mathcal{O})}$

$$QCoh(BKer) \simeq Mod_{R\Gamma(BKer, \mathcal{O})}$$

$$R\Gamma(BFix, \mathcal{O}) = R\Gamma(S^1, \mathbb{Z}_{(p)})$$

$$R\Gamma(BKer, \mathcal{O}) = \underbrace{\mathbb{Z}_p \oplus \mathbb{Z}_p[-1]}_{\text{square zero}}$$

as $\mathbb{E}_1$-rings they are same

$$\Rightarrow \quad QCoh(BFix) = QCoh(BKer)$$